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Functional BES equation

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Abstract: We give a realization of the Beisert, Eden and Staudacher equation for the planar $\mathcal{N}=4$ supersymetric gauge theory which seems to be particularly useful to study the strong coupling limit. We are using a linearized version of the BES equation as two coupled equations involving an auxiliary density function. We write these equations in terms of the resolvents and we transform them into a system of functional, instead of integral, equations. We solve the functional equations perturbatively in the strong coupling limit and reproduce the recursive solution obtained by Basso, Korchemsky and Kotański. The coefficients of the strong coupling expansion are fixed by the analyticity properties obeyed by the resolvents.

Keywords: AdS-CFT Correspondence, Bethe Ansatz.

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1. Introduction and overview

One of the most exciting discoveries in the last few years is the integrability of the maximally supersymmetric Yang-Mills theory [1 – 3] and its relation to the superstring theory in $AdS^5 \times S^5$ background [4 – 6]. An all-orders version of the Bethe Ansatz equations for AdS/CFT, valid asymptotically, was first formulated by Beisert, Eden and Staudacher [7], based on previous work [8 – 10]. The proposal by Beisert, Eden and Staudacher solves the crossing symmetry conditions, formulated by Janik [11]. This proposal, interpolating all the way from strong to weak coupling, was intensively tested. The most sophisticated tests were performed on the so-called twist-two anomalous dimension. In the limit of large Lorentz spin S, this quantity scales logarithmically [12 – 16]

$$\Delta - S = f(g) \ln S + \dots \,, \tag{1.1}$$

where g is the coupling constant, related to the 't Hooft coupling constant λ by

$$g^2 = \frac{\lambda}{16\pi^2} \,. \tag{1.2}$$

The universal scaling function $f(g)$ was computed perturbatively in the gauge theory up to the fourth order. The third order result was extracted [17] from a QCD computation by Moch, Vermaseren and Vogt [18]. The universal scaling function appears in the iterative structure of the multigluon amplitude [19] and it was computed to the third odrer in [20] and numerically to the fourth order, after an impressive effort [21, 22]. On the string side, the universal scaling function was also computed for the first three non-trivial orders [23 – 26]

$$f(g) = 4\,g - \frac{3\,\log 2}{\pi} - \frac{\mathrm{K}}{4\,\pi^2}\frac{1}{g} + \dots \,, \tag{1.3}$$

where K$= \beta(2)$ is Catalan's constant. It is remarkable that both the weak coupling and the strong coupling results for the universal scaling function can be reproduced from the conjectured Bethe ansatz equations. In this context, it is determined by the integral equation, written down by Eden and Staudacher [15]

$$\sigma(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} dv\, \frac{\sigma(v)}{(u-v)^2+1} - \int_{-\infty}^{\infty} dv\, K(u,v)\, \left(\sigma(v) - \frac{1}{4\pi g^2}\right) \,. \tag{1.4}$$

With the integration kernel $K(u, v)$ determined in [7], this equation is known as the Beisert, Eden and Staudacher (BES) equation. The universal scaling function is given by the integral of the density [27]

$$f(g) = 16 g^2 \int \sigma(u)\, du \,. \tag{1.5}$$

Although it is not, at least for the moment, possible to solve the equation (1.4) in a closed form for arbitrary g, it is relatively easy to extract from it the perturbative expansion at weak coupling [15, 7]. These coefficients of the perturbative expansion agree with the field-theoretic results [17, 20 – 22].

The strong coupling limit of the equation (1.4) proved to be much more difficult to master analytically. The first results were obtained numerically [28] and they correspond to the three coefficients in (1.3). The first coefficient in (1.3) was obtained analytically by various methods [27, 29 − 31], while the second was obtained, although not from the BES equation, by Casteill and Kristjansen [32] and later by Belitsky [33]. The reason the strong coupling limit of the BES equation is so difficult to take is that the expansion of the scattering phase in powers of $1/g$ is not uniform in the rapidity variable u. There are three different regimes for u which are to be considered. The first is the plane-wave limit [34], where $|u/2g| \gtrsim 1$, or in terms of momenta $p \sim 1/g$. The second is the so-called giant magnon regime [35], with $|u/2g| \lesssim 1$ or $p \sim 1$. The third regime was called [36] the near-flat space regime, at it correspond to to $u \pm 2g \sim 1$, or to momenta of the order $p \sim 1/\sqrt{g}$. As Maldacena and Swanson pointed out in [36], in this region the perturbative expansion of the dressing phase is completely reorganized, compared to that in plane-wave and giant magnon regimes. This is the reason why the attempts to to solve the BES equation order by order in $1/g$ failed beyond the leading order[1] .

An important step forward was made recently by Basso, Korchemsky and Kotański [37], who succeeded to give a procedure for obtaining all orders in the strong coupling expansion recursively. One of the important ingredients of their work was to linearize the Bethe ansatz equations by transforming the BES equation into a set of two equations, by exploiting the expression of the dressing kernel as a convolution of two "undressed" kernels [7]. The result is a set of two coupled integral equation, for the physical and an auxiliary density. The idea of linearization was first proposed by Kotikov and Lipatov [27] and subsequently by Eden [38]. Basso, Korchemsky and Kotański [37] used the Fourier representation of the BES equation, which is more adapted for numerical analysis, as well as a number of numerically inspired hypotheses.

The present work arose as an attempt to derive the results of [37] by purely analytical consideration. A natural way of solving the BES equation is to reformulate it as a functional equation for the resolvent. Here, we will pursue this direction. After writing the linearized BES equations in terms of the resolvent, it is possible to transform the integral BES equations into a set of functional equations for the physical and auxiliary resolvents. This procedure works for an arbitrary value of the coupling constant.[2]

The question whether the functional BES equations can be solved for any value of the coupling constant is still open. What is clearly possible to do is to solve these equations perturbatively at strong coupling, that is, by neglecting the non-perturbative corrections. This is possible because, in the absence of non-perturbative terms, the resolvents posses extra symmetry properties. The algorithm of solving the equations loosely follows the one of Basso, Korchemsky and Kotański [37].

First, we find the general solution for the resolvents at strong coupling in the giant magnons/plane-wave regimes. In these regimes the rescaled rapidity $u/2g$ is kept finite.

[1] In the approaches [32, 33], which in principle treat a different limit than the BES equation, the contribution from the near-flat space regime is suppressed from the beginning. This rapidity regime falls inside the gap of the density.

[2] The essential steps of this procedure are already present in the paper [27] by Kotikov and Lipatov.

The functional equations are linear and homogeneous, so that the general solution is a linear combination of a countable set of particular solutions. Each of these functions have a non-integrable singularity at the points $u/2g = \pm 1$. The indeterminacy in the coefficients of the linear combination and the singularities of the individual solutions can be cured by analyzing the solution in the near-flat space region, where the perturbation series in $1/g$ is reorganized compared to the giant magnon and plane-wave regimes. In the near-flat space regime the resolvents must have a series of integrable square root singularities in the lower half-plane and must decrease as $1/u$ at infinity in the upper half-plane. Remarkably, the condition that the solutions in the plane-wave/giant magnon and near-flat space regimes match analytically allows to determine uniquely the resolvents in the strong coupling limit up to non-perturbative corrections. This matching condition is equivalent to the quantization condition by Basso, Korchemsky and Kotański [37].

We formulate a recursive procedure for computing analytically the coefficients of the $1/g$ expansion of the density of Bethe roots. In the plane-wave regime, $|u/2g| > 1$, the density has a fourth order branch point at $u/2g = \pm 1$ at any order in $1/g$.

The paper is organized as follows: in section 2 we derive the linearized equations for the resolvents, in section 3 we transform the integral equations into functional equations and we discuss the analyticity properties of the resolvents and in section 4 we find the perturbative solution by imposing the required analyticity properties to the general solution.

1.1 Notations

The notations used in this paper are similar with those of our previous paper [30]. We will denote by ϵ the inverse gauge coupling and will use a rescaled rapidity, more adapted to the strong coupling limit:

$$\epsilon \equiv \frac{1}{4g}, \qquad u = \frac{u_{\rm old}}{2g}\,, \tag{1.6}$$

as well as the variable $x(u)$, related to u by

$$u(x) \equiv \frac{1}{2}\left(x + \frac{1}{x}\right), \quad x(u) = u\left(1 + \sqrt{1 - \frac{1}{u^2}}\right)\,. \tag{1.7}$$

Note the branch cut of $x(u)$ for $u \in [-1, 1]$. In the intermediate steps we will also use the notations

$$x^{\pm}(u) \equiv x(u \pm i\epsilon)\,. \tag{1.8}$$

Sometimes it will be useful to switch to the parametrization which resolves the square root of $x(u)$ and which is the hyperbolic limit of the elliptic parametrization in [30]:

$$u = \coth s\,, \quad x(s) = \coth\frac{s}{2}, \qquad \frac{u+1}{u-1} = e^{2s}\,. \tag{1.9}$$

The BES equation is formulated for the density function $\sigma(u)$ in the limit of large spin $S \gg 1$, related to the distribution $\rho(u)$ of Bethe roots by

$$\rho(u) = 8g^2 \log S\left(\frac{2\epsilon}{\pi} - \sigma(u)\right) + \mathcal{O}(S^0)\,, \qquad |u| \ll S/g. \tag{1.10}$$

2. Linearized BES equations for the resolvent

In this section, we reformulate the Beisert, Eden and Staudacher equation [15, 7], which determines the density of Bethe roots corresponding to the twist-two operator, as a set of two equations for the physical resolvent and an auxiliary resolvent. This is essentially the program which was carried out by Kotikov and Lipatov [27] and by Eden [38], although they have not explicitly identified the inverse Fourier transform of the density on the positive half-axis as the resolvent.

2.1 The resolvent

Let us consider the density $\sigma(u)$, supported on the real axis, as well as its Fourier transform[3] $\sigma(t)$

$$\sigma(t) = \int_{-\infty}^{\infty} du \; e^{itu} \sigma(u) \; . \tag{2.1}$$

The resolvent is defined, as usually, by

$$R_{\rm phys}(u) = \int\limits_{-\infty}^{\infty} dv \; \frac{\sigma(v)}{u-v} \; . \tag{2.2}$$

and it can be seen as the inverse Fourier transform of the density $\sigma(t)$ on the positive half-axis. Let us assume that u is in the upper half plane. Then we can write

$$\begin{aligned} R_{\rm phys}(u) &= -i \int_{-\infty}^{\infty} dv \int_0^{\infty} dt \; e^{it(u-v)} \sigma(v) & (2.3) \\ &= -i \int_0^{\infty} dt \; e^{itu} \sigma(-t) \, . & (2.4) \end{aligned}$$

The density is given by the discontinuity of the resolvent across the real axis:

$$\sigma(u) = \frac{1}{2\pi i} \left[R_{\rm phys}(u - i0) - R_{\rm phys}(u + i0) \right] \tag{2.5}$$

Since the density is supported by the whole real axis, the resolvent is given by two different analytic functions in the upper and in the lower half-planes. Due to the symmetry $\sigma(-u) = \sigma(u)$ we have the relation

$$R_{\rm phys}(-u) = -R_{\rm phys}(u) \tag{2.6}$$

Therefore, the symmetry determines the resolvent in the lower half plane once the resolvent in the upper half plane is known. The behavior of the resolvent at infinity is related to the universal scaling function

$$R_{\rm phys}(u) \sim \frac{1}{u} \int\limits_{-\infty}^{\infty} dv \, \sigma(v) = \frac{1}{u} \, \frac{f(g)}{16 g^2} \; . \tag{2.7}$$

[3] Our definition of $\sigma(t)$ is slightly different of that of [15] and the subsequent references.

2.2 Linearizing the BES equation

The linearization of the BES equation is best performed on the Fourier transformed form, although the manipulations can be done abstractly without reference to a particular representation. In our notations, the Fourier transformed BES equation reads

$$(1-e^{-2\epsilon t})\sigma(t) = -\int_0^\infty \frac{dt'}{2\pi}\left(K(t,t') + K_d(t,t')\right)\left(\sigma(t') - \sigma_0(t')\right) . \tag{2.8}$$

where $\sigma_0(t) = 4\epsilon\delta(t)$. Here we use the conventions

$$K(t,t') = 2\pi t e^{-(t+t')\epsilon} \sum_{n>0} 2n \frac{J_n(t)J_n(t')}{tt'} \equiv K_+(t,t') + K_-(t,t') , \tag{2.9}$$

where K_+ and K_- contain the expansion on *odd* and *even* order Bessel functions respectively, and the dressing kernel is given by the "magic formula" [7]

$$K_d(t,t') = 2\int_0^\infty \frac{dt''}{2\pi} K_-(t,t'') \frac{1}{1-e^{-2\epsilon t''}} K_+(t'',t') . \tag{2.10}$$

Two representations of the dressing kernel in the rapidity space were given in [30] and [39]. It is convenient to introduce an operator S, diagonal in Fourier representation:

$$S(t) = \frac{1}{1-e^{-2\epsilon t}} . \tag{2.11}$$

The equation (2.8) can be written symbolically as

$$-2\sigma = [(1+2SK_-)(1+2SK_+) - 1](\sigma - \sigma_0) . \tag{2.12}$$

Now it is possible to transform the BES equation into a pair of equations with the "main" kernels $K_\pm$ appearing linearly. This can be done at the expense of introducing an auxiliary density τ defined by

$$\tau + \sigma_0 \equiv -(1+2SK_+)(\sigma - \sigma_0) . \tag{2.13}$$

The linear system of equations obeyed by the physical and auxiliary density σ and τ is simply

$$\begin{aligned} \tau + \sigma &= -2SK_+(\sigma - \sigma_0) \\ \tau - \sigma &= -2SK_-(\tau + \sigma_0) , \end{aligned} \tag{2.14}$$

or, in Fourier representation,

$$\begin{aligned} (1-e^{-2\epsilon t})(\tau(t) + \sigma(t)) &= -2\int_0^\infty \frac{dt'}{2\pi} K_+(t,t')\,(\sigma - \sigma_0)(t') , \\ (1-e^{-2\epsilon t})(\tau(t) - \sigma(t)) &= -2\int_0^\infty \frac{dt'}{2\pi} K_-(t,t')\,\tau(t') . \end{aligned} \tag{2.15}$$

In the last line we have used that $K_-(t,0) = 0$.

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2.3 Holomorphic BES kernels

The next step is to transform back the equations (2.15) in the rapidity space. As we mentioned above, the inverse half-space Fourier transform of the density $\sigma(t)$ gives the resolvent $R_{\rm phys}(u)$. The latter defines a pair of functions $R^{\rm up}(u)$ and $R^{\rm down}(u) = -R^{\rm up}(-u)$, analytic respectively in the upper and lower rapidity half-planes. Because of the symmetry property, we are going in the following to concentrate exclusively on $R^{\rm up}(u)$. It is this function, together with its analytical continuation beyond the real axis, which will be denoted in the following by $R_{\rm phys}(u)$.

Assume that $\Im u > 0$ and perform the half-space inverse Fourier transformation to the rapidity plane,

$$\int_0^\infty \frac{dt}{2\pi} e^{itu} \int_0^\infty \frac{dt'}{2\pi} K(t,t') f(t') = \int_{-\infty}^\infty dv \int_0^\infty \frac{dt}{2\pi} e^{itu} \int_0^\infty \frac{dt}{2\pi} e^{-it'v} K(t,t') \int_0^\infty \frac{dt''}{2\pi} e^{it''v} f(t'') \,. \quad (2.16)$$

Therefore, since we intend to work only with functions defined in the upper half plane, we can retain only half of the original kernel in rapidity space, namely

$$K^\epsilon(u,v) = \int_0^\infty \frac{dt}{2\pi} \int_0^\infty \frac{dt'}{2\pi} e^{itu - it'v} K(t,t') \,. \quad (2.17)$$

Here we use the superscript ϵ for the kernel in order to indicate that it depends on the coupling constant $g = 1/4\epsilon$. Explicitly the "holomorphic" part of the odd and the even kernels reads

$$\begin{aligned} K^\epsilon_-(u,v) &= -\frac{1}{2\pi i}\frac{d}{du}\left[\ln\left(1 - \frac{1}{x^+ y^-}\right) + \ln\left(1 + \frac{1}{x^+ y^-}\right)\right] \\ K^\epsilon_+(u,v) &= -\frac{1}{2\pi i}\frac{d}{du}\left[\ln\left(1 - \frac{1}{x^+ y^-}\right) - \ln\left(1 + \frac{1}{x^+ y^-}\right)\right] \,. \end{aligned} \quad (2.18)$$

The dependence on ϵ in (2.18) comes only through the shifts $x^\pm = x(u \pm i\epsilon)$ and it will be removed by change of variable and shift of the integration contour. The $\epsilon \to 0$ limit of the kernels (2.18) will be denoted without superscript

$$K_\pm(u,v) = \frac{1}{2\pi i}\frac{2}{1-x^2}\left(\frac{1}{y - \frac{1}{x}} \pm \frac{1}{y + \frac{1}{x}}\right) \,. \quad (2.19)$$

When using these equations, one has to remember that $\Im u > 0$ and $\Im v < 0$, or

$$x = x(u + i0), \quad y = y(v - i0) = 1/y(v + i0) \,. \quad (2.20)$$

2.4 BES equations for the resolvents

For later convenience, we introduce the shifted resolvents

$$\begin{aligned} R(u) &= -i\int_0^\infty dt\, e^{iut}\, e^{\epsilon t}\, \sigma(t) = R_{\rm phys}(u - i\epsilon) \\ H(u) &= -i\int_0^\infty dt\, e^{iut}\, \epsilon^{\epsilon t}\, \tau(t) \,. \end{aligned} \quad (2.21)$$

This definition is valid for $\Im[u] > 0$ and the first singularity for $R(u)$ and $H(u)$ is situated on the real axis. R and H can be analytically continued to $\Im[u] < 0$. We will introduce another pair of functions by

$$R_\pm(u) = \tfrac{1}{2}[R(u) \pm H(u)] \tag{2.22}$$

as well as the related functions $r_\pm(u)$

$$r_\pm(u) = R_\pm(u) - R_\pm(u + 2i\epsilon)\,. \tag{2.23}$$

Now we can take the inverse Fourier transform of the equations (2.15) and make the shifts $u \to u - i\epsilon$ and $v \to v + i\epsilon$, in order to get rid of the ϵ-dependence in the kernels $K^\epsilon_\pm(u, v)$. We also use that in the rapidity space the operator S is expressed in terms of the shift operator $D = e^{2i\epsilon\partial_u}$:

$$S^{-1} = 1 - D\,, \quad \text{where} \quad Df(u) = f(u + 2i\epsilon)\,. \tag{2.24}$$

We obtain the equations[4]

$$\begin{aligned} R_+(u) - R_+(u + 2i\epsilon) &= \frac{4i\epsilon}{x^2 - 1} - \int dv\; K_+(u, v)\,[R_+(v + 2i\epsilon) + R_-(v + 2i\epsilon)] \\ R_-(u) - R_-(u + 2i\epsilon) &= \int dv\; K_-(u, v)\,[R_+(v + 2i\epsilon) - R_-(v + 2i\epsilon)]\,. \end{aligned} \tag{2.25}$$

After the change of variables the integration contour for v goes along the shifted real axis $\mathbb{R} - i\epsilon$, but it can be placed anywhere between the branch cut $[-1, 1]$ of the kernels $K_\pm$ and the branch cut $[-1 - 2i\epsilon, 1 - 2i\epsilon]$ of the resolvent $R(u)$. We will assume that the contour is just below the real axis. The variable u originally lies in $\Im u > 0$, but we can analytically continue it to the whole rapidity plane, using that the integration kernels are holomorphic.

3. Functional equation

The kernels (2.19) look almost like Cauchy kernels, if not for the branch cut of the variables $x(u)$ and $y(v)$. This suggests that we may simplify the BES equation further. This can be done provided that we know the analytical properties of the functions the kernels act on. In this section, we derive the analytic properties of the resolvents $R_\pm(u)$ and of the functions $r_\pm(u)$ and translate the action of the kernel in terms of an integral on the interval $[-1, 1]$. This transformation will allow to transform the integral equations into a functional equation.

3.1 Analytic properties of the resolvents

We start with the linearized BES equations

$$\begin{aligned} r_+(u) &= \frac{4i\epsilon}{x^2 - 1} - \int_{\mathbb{R}-i0} dv K_+(u, v)\,[R_+(v + 2i\epsilon) + R_-(v + 2i\epsilon)] \\ r_-(u) &= \int_{\mathbb{R}-i0} dv K_-(u, v)\,[R_+(v + 2i\epsilon) - R_-(v + 2i\epsilon)] \end{aligned} \tag{3.1}$$

[4]This way of rewriting the BES equation have been first proposed by Kotikov and Lipatov in [27].

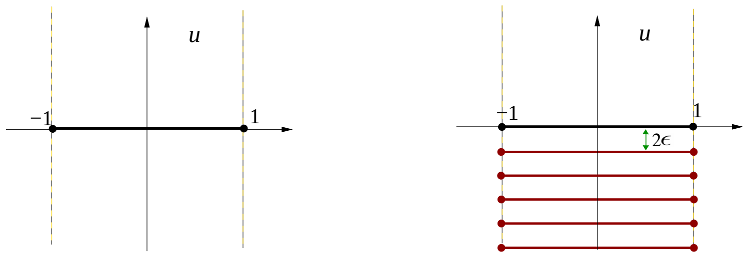

Figure 1: Left: Physical sheet for $r_\pm(u)$. Right: Physical sheet for $R_\pm(u)$.

where

$$K_\pm(u,v) = \frac{1}{2\pi i}\frac{2}{1-x^2}\left(\frac{1}{y-\frac{1}{x}} \pm \frac{1}{y+\frac{1}{x}}\right) \tag{3.2}$$

with $x = x(u)$ and $y = x(v)$. The variables u and v belong to the physical sheet, which means that $|x| > 1$ and $|y| > 1$. Since the kernel becomes singular only for u in the interval $[-1, 1]$, and there is no other singularity when u and v are on the physical sheet, the functions $r_\pm(u)$ are analytic in $\mathbb{C}\backslash[-1, 1]$.

We deduce that the resolvents

$$R_\pm(u) = \sum_{n=0}^{\infty} r_\pm(u + 2in\epsilon) \tag{3.3}$$

have a semi-infinite set of equidistant cuts as shown in figure 1. From the explicit form of the kernels it follows that

$$r_+(u) \propto \frac{1}{u^2}\,, \qquad r_-(u) \propto \frac{1}{u^3} \qquad (u \to \infty)\,. \tag{3.4}$$

The large u behavior of the resolvents is $R_+(u) \propto 1/u$ and $R_-(u) \propto 1/u^2$.

3.2 Analytic properties of the kernels

3.2.1 Changing the definition of the integration kernels

We will see that the integration kernels in (3.1) simplify significantly when acting on functions which are analytic in the upper half plane and on the real axis. Our strategy will be to use this simplified form and to extend it to functions that can have cuts on the real axis. Consider the integral

$$\int_{-\infty-i0}^{+\infty-i0} dv\ K_\pm(u,v)F(v)\,, \tag{3.5}$$

where $F(v)$ stands either for $R_+(v+2i\epsilon)$ or for $R_-(v+2i\epsilon)$. The function $F(u)$ is analytic in the half-plane $\Im u > -\epsilon$. We will actually need a weaker assumption, namely that the function $F(u)$ is analytic in the upper half plane $\Im u \geq 0$ with the real axis included.

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We would like to place the integration contour above the real axis, since in the upper half-plane both $K_\pm$ and F are analytic. Using the properties

$$\begin{aligned} x(v-i0) &= 1/x(v+i0)\ ,\ u\in[-1,1] \\ x(v-i0) &= x(v+i0)\ ,\ u\in\mathbb{R}\backslash[-1,1] \end{aligned} \tag{3.6}$$

we can place the integration contour above the real axis at the price of changing the form of the kernel in the interval $[-1,1]$. Next, since in the upper half-plane the function F is analytic, we can deform the rest of the contour to a contour that goes along the interval $[-1,1]$ in the opposite direction. Adding up the two contributions we evaluate the integral as

$$\begin{aligned} K_\pm F(u) &= \frac{2}{1-x^2}\int\limits_{-1+i0}^{1+i0}\frac{dv}{2\pi i}F(v)\left(\frac{-yx}{y-x}\pm\frac{yx}{y+x}-\frac{1}{y-\frac{1}{x}}\mp\frac{1}{y+\frac{1}{x}}\right) \\ &= \int\limits_{-1+i0}^{1+i0}\frac{dv}{2\pi i}F(v)\frac{y-\frac{1}{y}}{x-\frac{1}{x}}\left(\frac{1}{v-u}\mp\frac{1}{v+u}\right)\,. \end{aligned} \tag{3.7}$$

In the original equation (2.25), the integration contour is pinched between two cuts distanced by ϵ and the limit $\epsilon\to 0$ is not well defined. On the contary, the form (3.7) of the integration kernel has a smooth $\epsilon\to 0$ limit. As the function $F(u)$ is analytic on the real axis, we can write the above integral as

$$K_\pm F(u) = \int\limits_{-1}^{1}\frac{dv}{2\pi}\sqrt{\frac{1-v^2}{u^2-1}}\,\frac{F(v+i0)\pm F(-v+i0)}{v-u}\,. \tag{3.8}$$

We will use this expression as the definition of the action of the kernels on functions which have a cut on the interval $[-1,1]$. As a consequence, the necessary and sufficient condition that the function F is annihilated by $K_\pm$ is

$$K_\pm F=0 \quad\Leftrightarrow\quad F(u+i0)\pm F(-u+i0)=0\ ,\ u\in[-1,1]\,. \tag{3.9}$$

The last condition can be written in terms of the variable x as

$$F(x) = \mp F(-1/x)\,. \tag{3.10}$$

Note that the condition (3.9) does not imply the function $F(u)$ is odd or even.

3.2.2 Projective properties of the kernels $K_\pm$

Via the transformation $v\to -v$ the integral in (3.8) can be written as an integral over a closed contour around the segment $[-1,1]$:

$$K_\pm F(u) = \oint\frac{dv}{2\pi i}\ \tilde{F}(v)\sqrt{\frac{v^2-1}{u^2-1}}\,\frac{1}{v-u} \tag{3.11}$$

with

$$\tilde{F}(u) = \begin{cases} F(u) & \text{if } \Im u > 0\,, \\ \pm F(-u) & \text{if } \Im u < 0\,. \end{cases} \tag{3.12}$$

Denote by $\mathcal{L}^{\pm}$ the linear space of even/odd functions, analytic outside the interval $[-1,1]$ and decreasing at infinity faster than $1/u$ at infinity. For any $f_{\pm} \in \mathcal{L}^{\pm}$, the kernel $K_{\pm}$ acts as the identity operator:

$$K_{\pm} f_{\pm} = f_{\pm}, \quad f_{\pm} \in \mathcal{L}^{\pm}\,. \tag{3.13}$$

In particular,

$$K_{+} r_{+} = r_{+}, \quad K_{-} r_{-} = r_{-}\,. \tag{3.14}$$

Now consider the result of the action of the kernel $K_{\pm}$ on an arbitrary function $F(u)$. Since the kernel $K_{\pm}(u,v)$ is even/odd as a function of u, is analytic outside the interval $[-1,1]$, and decreases as $1/u^2$ at infinity, the resulting function $F_{\pm} = K_{\pm} F$ belongs to $\mathcal{L}^{\pm}$. As a consequence, $K_{\pm}^2 F = K_{\pm} F_{\pm} = F_{\pm} = K_{\pm} F$. We conclude that the kernel $K_{\pm}$ is idempotent:

$$K_{\pm}^2 = K_{\pm}\,. \tag{3.15}$$

An example of a function that does not belong to $\mathcal{L}^{\pm}$ is the constant function. The action of $K_{\pm}$, evaluated by expanding the contour to infinity, is

$$K_{+} \cdot 1 = -\frac{1/x}{\sqrt{u^2-1}} = \frac{2}{1-x^2}, \quad K_{-} \cdot 1 = 0\,. \tag{3.16}$$

The first equation (3.16) means that $f_0(x) = \frac{1+x^2}{1-x^2}$ is a zero mode of K_{+}. This function satisfies the condition (3.10): $f_0(-1/x) = -f_0(x)$.

3.3 The BKK transformation

We have seen that the integration kernels $K_{\pm}$ on the r.h.s. of the linearized BES equations (2.25) can be also defined by (3.8). The new definition coincides with the original one for the functions analytic in the upper half plane and on the real axis. From now on we will adopt the (3.8) of the integration kernels.

The linearized BES equations (2.25) considerably simplify when written in terms of the functions Γ_{+} and Γ_{-} defined by[5]

$$\begin{aligned} \Gamma_{+}(u) + \Gamma_{-}(u) &\equiv R_{+}(u) + R_{-}(u+2i\epsilon) + 2i\epsilon \\ \Gamma_{+}(u) - \Gamma_{-}(u) &\equiv R_{-}(u) - R_{+}(u+2i\epsilon) + 2i\epsilon\,. \end{aligned} \tag{3.17}$$

Indeed, with the help of the identities (3.16) and (3.14) we write the linearized BES equations (2.25) as

$$\begin{aligned} K_{+}(\Gamma_{+} + \Gamma_{-}) &= 0, \\ K_{-}(\Gamma_{+} - \Gamma_{-}) &= 0\,. \end{aligned} \tag{3.18}$$

[5]This transformation corresponds to the one given by eq. (6) in [37].

Therefore the solution of the BES equation is a linear combination of the zero modes of the operators $K_\pm$.

3.4 From integral to functional equations

Now we can reformulate the homogeneous integral equations (3.18) as a pair of functional equations for Γ_+ and Γ_-. According to (3.9) or (3.10), these equations imply the following boundary conditions on the upper edge of the cut $[-1,1]$,

$$\Gamma_+(u+i0)+\Gamma_-(-u+i0)=0,\quad u\in[-1,1]\,, \tag{3.19}$$

or, in terms of the variable x,

$$\Gamma_+(-1/x)=-\Gamma_-(x)\,. \tag{3.20}$$

The last equation should hold on the arc $|x|=1, \Im x>0$.

Hence the solution of the BES equation must be among the solutions of the functional relation. Note that this equation is exact in the sense that in the derivation we did not assumed that ϵ is small.

Of course this relation has a huge set of solutions. The physical solution is distinguished by imposing its analyticity properties in the vicinity of the singular points $u\to\infty$ and $u=\pm1$. The extra conditions that single out the physical solution are formulated in terms of the original resolvents $R_\pm=\frac{1}{2}(R\pm H)$, or equally the functions $r_\pm(u)$ defined by (2.23). Namely, the functions $r_\pm$ must be analytic everywhere outside the cut $[-1,1]$, where they have square-root singularities, and behave at infinity according to (3.4). This conditions determine the analytic properties of $\Gamma_\pm$, which are related to $r_\pm$ by

$$\Gamma_-(u)=\tfrac{1}{2}r_+(u)-\tfrac{1}{2}r_-(u)+\sum_{n=1}^{\infty}r_+(u+2in\epsilon)\,, \tag{3.21}$$

$$\Gamma_+(u)=2i\epsilon+\tfrac{1}{2}r_+(u)+\tfrac{1}{2}r_-(u)+\sum_{n=1}^{\infty}r_-(u+2in\epsilon)\,. \tag{3.22}$$

Since the functions $r_\pm$ have a square root cut along the interval $[-1,1]$ of the physical sheet, all singularities of the functions $\Gamma_\pm$ are of square root type.

4. Perturbative solution at strong coupling

In this section we obtain the perturbative solution for the resolvent. First we consider the limit $\epsilon\to0$ with u fixed. This limit corresponds to the plane waves (PW) or giant magnons (GM) regimes, depending on the interval where the rapidity takes its values (figure 2). The distribution of Bethe roots in the PW and the GM regimes is given by two different analytical expressions, but for the resolvent they are related by analytical continuation.

It happens that in the strong coupling limit, and in all orders in ϵ, the intricate cut structure of the resolvent and the related functions can be replaced by a single cut $u\in[-1,1]$, but with fourth order instead of second order branch points at $u=\pm1$.

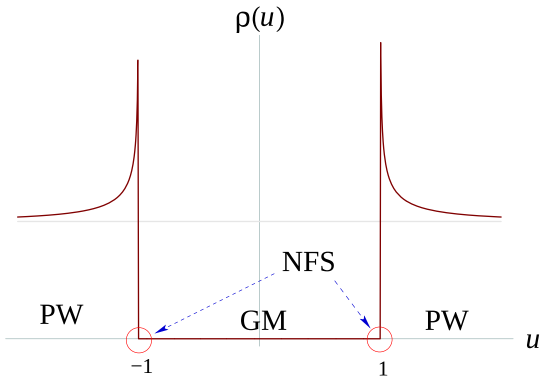

Figure 2: The physical density $\rho(u) = 2\epsilon/\pi - \sigma(u)$ in the strong coupling limit and the three regimes: plane waves (PW) for $u < -1$ and $u > 1$, giant magnons (GM) for $-1 < u < 1$, and near flat space (NFS) in the vicinity of the points $u = \pm 1$.

Furthermore, an important simplification stems from the fact that in the PW/GM regime the combinations $\Gamma_\pm(u)$ have definite parity,

$$\Gamma_\pm(-u) = \pm\Gamma_\pm(u)\,. \tag{4.1}$$

This will allow us to write the general solution of the linearized BES equation.

Since the equations are homogeneous, the general solution is a linear combination of all particular solutions with arbitrary coefficients c_n, which are functions of the coupling constant. The behavior of the resolvent at $u \to \infty$ gives one linear constraint on the coefficient functions $c_n(\epsilon)$, which is not sufficient to determine them.

The rest of the information is supplied by the conditions on the analytic propertirs of the solution in the vicinity of the singular points $u = \pm 1$. For this purpose we blow up the vicinity of the the two singular points so that the cut structure of the resolvent reappears. Instead of keeping u fixed, we take the limit $\epsilon \to 0$ either with $z = (u-1)/2\epsilon$ fixed or with $\bar{z} = -(u+1)/2\epsilon$ fixed. This strong coupling limit corresponds to the near flat space (NFS) regime [36]. Then we compare the power series expansion at $z = 0$, which follows from the analytic structure of the exact solution, with the expansion at $z = \infty$, which is determined by the perturbative solution in the PW/GM regime. The requirement that the two expansions match with each other is sufficient to determine both of them, order by order in ϵ. Technically it is more advantageous to compare the inverse Laplace transforms for which the shift operator D becomes diagonal. A recurrence procedure, analogous to that of [37], allows to obtain analytically the density of Bethe roots in any order in ϵ, both in the PW/GM and NFS regimes. We check that the result of [37] for the universal scaling function is correctly reproduced.

4.1 General form of the solution in the PW/GM regime

Let us first prove the symmetry property (4.1). For that we use the expression of $\Gamma_\pm$ in terms of the functions of definite parity $r_\pm(u) = \pm r_\pm(-u)$, given by (3.21) and (3.22). We observe that the combinations

$$\Gamma_\pm(u) \mp \Gamma_\pm(-u) \,=\, \sum_{n\in\mathbb{Z}} r_\mp(u+2in\epsilon) \tag{4.2}$$

are periodic functions with period $2i\epsilon$. From here and from the fact that $r_\pm(u) \sim 1/u^2$ at infinity it follows that the r.h.s. (4.2) vanishes in the limit $\epsilon \to 0$ up to non-perturbative terms. To see that we perform Poisson resummation. Assuming that $\Re u > 1$, we have

$$\sum_{n=-\infty}^{\infty} r_\pm(u+2in\epsilon) \,=\, \frac{1}{\epsilon}\sum_{n=1}^{\infty} e^{-\pi n u/\epsilon} \oint \frac{dv}{2\pi i} e^{\pi n v/\epsilon} r_\pm(v)\,, \tag{4.3}$$

where the integration contour closes around the physical cut $[-1,1]$ of $r_\pm$. The series in $e^{-\pi n u/\epsilon}$ is rapidly convergent when $u > 1$ and diverges at $u = 1$. When $\Re u < -1$ we get a similar expansion, but the opposite sign in the exponents. In both cases the result is exponentially small except at the points $u = \pm 1$.

Therefore, if we neglect these non-perturbative corrections, the solution should have the additional symmetry (4.1). Then the functional equation (3.19) can be replaced by a simpler one,

$$\begin{aligned} \Gamma_+(u+i0) \,&=\, +\Gamma_-(u-i0) \\ \Gamma_+(u-i0) \,&=\, -\Gamma_-(u+i0)\,. \end{aligned} \tag{4.4}$$

We remind that these equations are valid on the cut, where $u \in [-1,1]$. From here it follows that $\Gamma_\pm$ are obtained as different branches of the same meromorphic function, defined on a four-sheet Riemann surface.

In terms of the global parameter of the Riemann surface, s, the functional equations get the form of periodic conditions

$$\Gamma_\pm(s \pm i\pi) \,=\, \pm\Gamma_\mp(s)\,. \tag{4.5}$$

It is convenient to work with the combinations

$$G_\pm = \Gamma_+ \pm i\Gamma_-\,, \tag{4.6}$$

for which (4.1) and (4.5) take the form

$$G_\pm(-s) = G_\mp(s)\,, \quad G_\pm(s+i\pi) = \pm i G_\pm(s)\,. \tag{4.7}$$

We can represent the general solution in the form of the series:

$$\begin{aligned} G_\pm(s) \,&=\, 2i\epsilon \sum_{n\in\mathbb{Z}} c_n(\epsilon)\, e^{\pm(2n+1/2)s} \\ &=\, 2i\epsilon \sum_{n\in\mathbb{Z}} c_n(\epsilon) \left(\frac{u+1}{u-1}\right)^{\pm n \pm \frac{1}{4}}\,. \end{aligned} \tag{4.8}$$

We will first obtain the general form of the solution in the PW/GM regime. In this regime the branch points condense into continuous lines starting at the points $u = \pm 1$ and the resolvents are described, as we will see later, by meromorphic functions with a single pair of branch points at $u = \pm 1$.

The perturbative solution (4.8) is valid in the limit where the distance between the subsequent branch points vanishes and the infinite sequence of simple branch points starting at $u = \pm 1$ produces a fourth order branch singularity at $u = \pm 1$.

The solution has three singular points, $u = 1, u = -1$ and $u \to \infty$. As usual in such kind of problems, the coefficients functions $c_n(\epsilon)$ in the series (4.8) will be evaluated by matching with the asymptotic behavior of the solution at the singular points.

4.1.1 The density in the GM regime

The general form of the solution given by (4.8) is sufficient to determine perturbatively the density in the giant magnon regime. Indeed, inspecting each of the terms, one can verify that the value of the resolvent on the interval $-1 < u < 1$, and therefore the density, is constant and is given by the leading order.

This fact is actually a direct consequence of the equations (4.5) and the symmetry (4.1). Indeed, using the anti-symmetry of the resolvent R_{phys}, we can express the fluctuation density in terms of the values of the resolvent above the real axis:

$$\sigma(u) = -\frac{1}{2\pi i}[R_{\text{phys}}(u + i0) + R_{\text{phys}}(-u + i0)] \tag{4.9}$$

$$= -\frac{1}{2\pi i}[R(u + i\epsilon) + R(-u + i\epsilon)]\,. \tag{4.10}$$

Further, by the definition (3.17), the resolvent R_{phys} is expressed in terms of $\Gamma_\pm$ as

$$R_{\text{phys}} = -2i\epsilon + \frac{2}{D + D^{-1}}\left(D^{\frac{1}{2}}\Gamma_- + D^{-\frac{1}{2}}\Gamma_+\right) \qquad (\Im u > 0)$$

where D is the shift operator defined by (2.24). Applying the functional equation (4.4), we see that all the terms on the r.h.s. of (4.9) except the constant term cancel and therefore to all orders in ϵ

$$\sigma(u) = 2\epsilon/\pi, \quad u \in [-1, 1]\,. \tag{4.11}$$

This means the distribution of Bethe roots has a gap on the interval $[-1, 1]$. The physical density (1.10), which gives the distribution of Bethe roots, vanishes to all orders in ϵ in the GM regime.

4.1.2 Expansion at $u = \pm 1$ and a scaling condition for the coefficients

Let us examine the behavior of the solution (4.8) near the singular points $u = \pm 1$. We mentioned that the strong coupling limit is not uniform in u. The strong coupling solution has different properties in the limit considered above,

$$\epsilon \to 0 \qquad \text{with } u \text{ fixed} \qquad \text{(PW/GM)}, \tag{4.12}$$

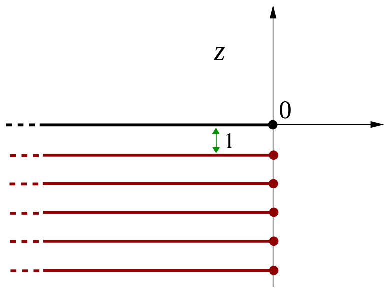

Figure 3: Physical sheet for $R_\pm(u)$ in the NFS limit.

and the limit

$$\epsilon \to 0 \qquad \text{with} \ \frac{u^2-1}{\epsilon} \ \text{fixed} \qquad (\text{NFS})\,. \tag{4.13}$$

The singular behavior at $u = \pm 1$ in the PW/GM limit is an artifact of the rescaled rapidity (1.6). If we take the NFS limit (4.13), the solution for the density must be integrable at $u = \pm 1$. It is obvious that the strong coupling expansions in the two limits do not match since the solution (4.8) gives non-integer powers of ϵ when considered near $u = \pm 1$.

Our analysis of the analytical properties of the solution allows us to determine its general form near $u = \pm 1$. The conditions that it goes smoothly into the solution (4.8) obtained for the rest of the complex plane will be used in the next section to fix the coefficients c_n.

The complex variables relevant for the vicinity of the points $u = 1$ and $u = -1$ are

$$z = \frac{u-1}{2\epsilon}, \qquad \bar{z} = -\frac{u+1}{2\epsilon}\,. \tag{4.14}$$

The variable z coincides, up to a shift by $2g$, with the original (before rescaling by $2g$) rapidity in the BES equations.

In the NFS limit the cuts become semi-infinite, with the branchpoints placed at at $z = 0, -i, -2i, \dots$, as shown in figure 3. The functions $r_\pm(z)$ have by construction an integrable square root singularity at $z = 0$. Therefore they can be expanded at small z as

$$r_\pm(z) = \sum_{n\ge 0} b_n^\pm(\epsilon)\, z^{n-1/2} + \sum_{n\ge 0} d_n^\pm(\epsilon)\, z^n \qquad (\,|z| < 1\,). \tag{4.15}$$

The compatibility of the expansions (4.15) and (4.8) imposes strong restrictions on the coefficient functions $c_n(\epsilon)$. In particular, each term of the expansion (4.8) must have a non-singular limit $\epsilon \to 0$ when expressed in terms of the variable z or $\bar{z}$. This means that the coefficients $c_n(\epsilon)$ must scale as $\epsilon^{|n|}$, so that their Taylor series have the form

$$c_n(\epsilon) = \epsilon^{|n|}\,\alpha_n(\epsilon), \qquad \alpha_n(\epsilon) = \sum_{p=0}^{\infty} \alpha_{n,p}\,\epsilon^p \qquad (n \in \mathbb{Z})\,. \tag{4.16}$$

We arrive at the following expression of the general solution in terms of the shifted rapidity variable z:

$$G_\pm(z) = 2i\epsilon \sum_{n\in\mathbb{Z}} \epsilon^{|n|}\, \alpha_n(\epsilon) \left(\frac{1+\epsilon z}{\epsilon z}\right)^{\pm n \pm \frac{1}{4}} . \tag{4.17}$$

The strong coupling expansion of $G_\pm$ with z kept fixed is different than the expansion of the solution with u fixed, (4.8). In particular, it contains fractional powers of ϵ. The resolution of this paradox is in the non-uniformity of the strong coupling expansion with respect to the rapidity variable u. Near the singular points $u = \pm 1$ the strong coupling expansion should be performed according to the prescription (4.13) and not (4.12). The series (4.17) should be understood as an expansion at large z, possibly asymptotic, of the true solution, whose small z expansion is given by (4.15). The compatibility of (4.17) and (4.15) is studied more easily for the inverse Laplace transforms. This will be done in the next section where we will see that demanding that the two series are compatible fixes uniquely the coefficients of both of them.

4.1.3 Expansion at $u = \infty$ and universal scaling function

By construction, the solution (4.8) expands at infinity as

$$G_\pm(u) = \sum_{n\geq 0} \frac{W_n^\pm}{u^n} \tag{4.18}$$

Comparing the series with the large u asymptotics (2.7) of the physical resolvent

$$R_{\rm phys} = -2i\epsilon + \frac{\sqrt{D}}{1+D^2}\left[(1-iD)G_+ + (1+iD)G_-\right] , \tag{4.19}$$

we fix the first two coefficients $\frac{1}{2}(W_0^+ + W_0^-) = 2i\epsilon$ and $\frac{1}{2}(W_1^+ + W_1^-) = f(g)/16g^2$. This yields a constraint for the expansion coefficients,

$$1 = \sum_{n\in\mathbb{Z}} c_n(\epsilon) \equiv \sum_{n\in\mathbb{Z}} \epsilon^{|n|}\, \alpha_n(\epsilon) , \tag{4.20}$$

and the expression of the universal scaling function $f(g)$ in terms of c_n:

$$\begin{aligned} f(g) &= \frac{1}{\epsilon}\sum_{n\in\mathbb{Z}}(4n+1)c_n = \frac{1}{\epsilon} + \frac{1}{\epsilon}\sum_{n\neq 0} 4nc_n \\ &= \frac{1}{\epsilon} + 4\sum_{n\neq 0}\epsilon^{|n|-1}\, n\, \alpha_n(\epsilon) . \end{aligned} \tag{4.21}$$

4.1.4 The leading order in the PW/GM limit

It follows from the scaling (4.16) that the solution at the leading order is given by the $n = 0$ term of the series

$$G_\pm(s) = 2i\epsilon \left(\frac{u+1}{u-1}\right)^{\pm\frac{1}{4}} . \tag{4.22}$$

The constraint (4.20) gives $c_0(0) = 1$ and the universal scaling is given by the $n = 0$ term in (4.21):

$$f(g) = \frac{1}{\epsilon} = 4g\,. \tag{4.23}$$

Written for the resolvent and in terms of the variable $x(u)$, the leading order solution (4.22) is

$$R_{\epsilon=0}(u) = -2i\epsilon\left(1 - \frac{1}{\sqrt{1-1/x^2}} + i\frac{1/x}{\sqrt{1-1/x^2}}\right). \tag{4.24}$$

The density $\sigma(u)$, related to the resolvent by (2.5), agrees with the AABEK solution [29, 30].

4.2 Inverse Laplace transform of the solution

The relation (4.2) involves the shift operator and therefore looks simpler for the Fourier transformed quantities. However, in order to be able to exploit the analytic properties of the general solution we perform instead an inverse Laplace transformation. Since the functions $g_\pm$ and $G_\pm$ are analytic for $\Re z > 0$, we can define the Laplace transformation and its inverse

$$f(z) = \int_0^\infty d\ell\, e^{-z\ell}\, \tilde{f}(\ell) \qquad \tilde{f}(\ell) = \frac{1}{2\pi i}\int_{i\mathbb{R}+0} dz\, e^{z\ell}\, f(z)\,. \tag{4.25}$$

Similarly we can define the inverse Laplace transformation for the variable $\bar{z}$ having as the origin the left branch point.

Introduce, similarly to (4.6), the linear combinations

$$g_\pm = r_+ \mp i r_-\,. \tag{4.26}$$

Then from (3.21) it follows that the functions $g_\pm(z)$ are related to $G_\pm$ to by

$$g_\pm = \frac{1 \pm i}{D \mp i}\,(D-1)\,G_\pm\,, \tag{4.27}$$

where $D = e^{i\partial_z}$ is the shift operator defined in (2.24). For the inverse Laplace images $\tilde{g}_\pm$ and $\tilde{G}_\pm$ this relation takes the form

$$\tilde{g}_\pm(\ell) = \frac{\sqrt{2}\sin(\frac{\ell}{2})}{\sin(\frac{\ell}{2} \pm \frac{\pi}{4})}\,\tilde{G}_\pm(\ell)\,. \tag{4.28}$$

Our aim is to use the relation (4.28) to investigate the compatibility of the general solution (4.8) with the expansion (4.15) at $z = 0$, which in the ℓ-space becomes expansion at $\ell \to \infty$:

$$\tilde{g}_\pm(\ell) = \ell^{-1/2}\sum_{n\geq 0}\tilde{g}_n^\pm\,\ell^{-n} + \sum_{n\geq 0}\tilde{h}_n^\pm\,\ell^{-n-1}\,. \tag{4.29}$$

It follows from the analytic properties of the resolvents in the rapidity space that, in the NFS limit, $\tilde{g}_\pm(\ell)$ are analytic everywhere except for the negative real axis, while $\tilde{G}_\pm(\ell)$

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are analytic everywhere on the positive real axis. We sketch the proof in appendix A. The explicit expression for the inverse Laplace transform of (4.17) is a series of confluent hypergeometric functions of the first kind

$$\tilde{G}_{\pm}(\ell) = \pm 2i \sum_{n\in\mathbb{Z}} \epsilon^{|n|}\alpha_n(\epsilon)\, \left(n+\tfrac{1}{4}\right)\, {}_1F_1\left(1\mp\tfrac{1}{4}\mp n; 2; -\ell/\epsilon\right)\,. \tag{4.30}$$

The PW/GM corresponds to keeping $\zeta\equiv\ell/\epsilon$ finite when $\epsilon\to 0$ while the NFS regime is obtained when keeping ℓ fixed. In the NFS limit we expand in ϵ with ℓ fixed. Therefore, in order to compare with (4.29), we are going to use the asymptotic expansion the limit $\ell/\epsilon\to\infty$, where the solution has an essential singularity:

$$\begin{aligned} {}_1F_1\left(a;2;-\ell/\epsilon\right) \propto \left(\ell/\epsilon\right)^{-a}\, {}_2F_0\left(a,a-1;;\ \epsilon/\ell\right)/\Gamma(2-a) \\ +e^{-\ell/\epsilon}\left(-\ell/\epsilon\right)^{a-2}\, {}_2F_0\left(1-a,2-a;;\ \epsilon/\ell\right)/\Gamma(a)\,. \end{aligned} \tag{4.31}$$

The asymptotic expansion of the inverse Laplace image of the solution in this limit is evaluated using by (4.31). As far as we interested in the perturbative solution, we can neglect the second exponentially small term in (4.31) and write

$$\tilde{G}_{\pm}(\ell) = 2i\sum_{n\in\mathbb{Z}}\epsilon^{|n|}\alpha_n(\epsilon)\frac{(\epsilon/\ell)^{1\mp\frac{1}{4}\mp n}}{\Gamma(\pm\frac{1}{4}\pm n)}\, {}_2F_0\left(1\mp\tfrac{1}{4}\mp n, \mp\tfrac{1}{4}\mp n;;\ \epsilon/\ell\right)\,. \tag{4.32}$$

In the leading order in ϵ

$$\tilde{G}_{\pm}(\ell) = 2i(\epsilon/\ell)^{1\mp\frac{1}{4}}\left(\sum_{n=0}^{\infty}\frac{\alpha_{\pm n,0}}{\Gamma(n\pm\frac{1}{4})}\,\ell^n + \mathcal{O}(\epsilon)\right)\,. \tag{4.33}$$

We see that even in the leading order the resolvents scale in the NFS regime as fractional powers of ϵ and are linked to the whole perturbative series in the PW/GM regime. In the leading order the sum on the r.h.s. of (4.33) contains only non-negative powers of ℓ, but in the next orders in ϵ more and more negative powers of ℓ will appear.

Now we represent, as in [37], the ratio of the sine functions in (4.28) as

$$\frac{\sin(\frac{\ell}{2})}{\sin(\frac{\ell}{2}\pm\frac{\pi}{4})} = \frac{S_{\pm}(\ell)}{T_{\pm}(\ell)}\,, \tag{4.34}$$

where S and T represent ratios of Gamma functions:

$$S_{\pm}(\ell) = \pm\frac{\Gamma(\frac{1}{2}+\frac{\ell}{2\pi}\mp\frac{1}{4})}{\Gamma(\frac{\ell}{2\pi})}\,, \qquad T_{\pm}(\ell) = \frac{\Gamma(1-\frac{\ell}{2\pi})}{\Gamma(\frac{1}{2}-\frac{\ell}{2\pi}\pm\frac{1}{4})}\,. \tag{4.35}$$

If we rewrite the equation (4.28) as

$$\frac{\tilde{G}_{\pm}(\ell)}{T_{\pm}(\ell)} = \frac{1}{\sqrt{2}}\,\frac{\tilde{g}_{\pm}(\ell)}{S_{\pm}(\ell)}\,, \tag{4.36}$$

then the l.h.s. is analytic everywhere except the negative real axis, while the r.h.s. is analytic everywhere except the positive real axis. As a consequence, neither of the sides has poles and the only singularities can be branch points at $\ell = 0$ and $\ell = \infty$. This means, in particular, that the expansion of the r.h.s. as a power series at $\ell = \infty$ coincides with the expansion of the l.h.s. at $\ell = 0$.

To evaluate the coefficients of the two power series we need to expand $S_\pm$ at $\ell = +\infty$ and $T_\pm$ at $\ell = 0$,

$$S_\pm(\ell) = \pm (\ell/2\pi)^{\frac{1}{2}\mp\frac{1}{4}} \left(1 + \sum_{n=1}^{\infty} S_n^\pm \ell^{-n}\right) , \tag{4.37}$$

$$T_\pm(\ell) = \frac{1}{\Gamma\left(\frac{1}{2} \pm \frac{1}{4}\right)} \left(1 + \sum_{n=1}^{\infty} T_n^\pm \ell^{n}\right) . \tag{4.38}$$

As it should, the series expansion of the l.h.s. of (4.36) at $\ell = 0$ contains exactly the same fractional powers as that for the r.h.s. at $\ell = \infty$. To get rid of these fractional powers, we multiply both sides of (4.36) by $(\ell/\epsilon)^{1\mp 1/4}$ and write

$$\frac{\tilde{G}_\pm(\ell)}{\tilde{T}_\pm(\ell)} (\ell/\epsilon)^{1\mp\frac{1}{4}} = \sum_{n\in\mathbb{Z}} C_n^\pm(\epsilon)\, \ell^{-n} , \tag{4.39}$$

where the coefficients $C_n^\pm(\epsilon)$ should be understood as formal series in ϵ,

$$C_n^\pm(\epsilon) = \sum_{p=0}^{\infty} C_{n,p}^\pm\, \epsilon^p . \tag{4.40}$$

From (4.29), (4.37) and the relation (4.36) we deduce that the coefficients in front of the non-negative powers of ℓ vanish,

$$C_n^\pm(\epsilon) = 0 \quad \text{for} \quad n = -1, -2, \dots \ . \tag{4.41}$$

Solving these contraints (4.20) and (4.41) order by order in ϵ one can evaluate recursively the Taylor coefficients $\alpha_n^\pm$ of the series (4.16). The recurrence procedure is possible because at each order in ϵ the sum on the r.h.s. of (4.39) contains only a finite number of negative powers of ℓ.

We have learned that the general solution of the BES equation in the NFS limit is of the form

$$\tilde{G}_\pm(\ell) = (\ell/\epsilon)^{-1\pm\frac{1}{4}}\, T_\pm(\ell) \sum_{n\geq 0} C_n^\pm(\epsilon)\, \ell^{-n} , \tag{4.42}$$

$$\tilde{g}_\pm(\ell) = \sqrt{2}\, (\ell/\epsilon)^{-1\pm\frac{1}{4}}\, S_\pm(\ell) \sum_{n\geq 0} C_n^\pm(\epsilon)\, \ell^{-n} , \tag{4.43}$$

with computable coefficient functions given by the formal Taylor series (4.40). Comparing the expansions of (4.32) and (4.42) at each order in ϵ and imposing the condition (4.20) we can evaluate both sets of coefficients $\alpha_{n,p}$ and $C_{n,p}^\pm$. We show below how the procedure works for the leading order.

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4.2.1 The leading order in the NFS limit

In the leading order in ϵ the series expansion of $\tilde{G}_{\pm}$ at $\ell = 0$, given by (4.33), contains only non-negative powers in ℓ. Therefore the sum on the r.h.s. of (4.39) contains only the term with $n = 0$, and we have

$$(\ell/\epsilon)^{1\mp\frac{1}{4}}\tilde{G}_{\pm}(\ell) = 2i\sum_{n=0}^{\infty}\frac{\alpha_{\pm n,0}}{\Gamma(n\pm\frac{1}{4})}\,\ell^n \; = T_{\pm}(\ell)\,C^{\pm}_{0,0}\,. \tag{4.44}$$

From the constraint (4.20), which in the leading order gives $\alpha_{0,0} = 1$, we evaluate

$$C^{+}_{0,0} = 2i\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\,,\quad C^{-}_{0,0} = 2i\frac{\Gamma(\frac{1}{4})}{\Gamma(-\frac{1}{4})}\,. \tag{4.45}$$

For the other coefficients we find

$$\alpha_{\pm n,0} = \frac{\Gamma(n\pm\frac{1}{4})}{\Gamma(\pm\frac{1}{4})}\,T^{\pm}_n\,, \tag{4.46}$$

where $T^{\pm}_n$ are the coefficients in the expansion (4.38),

$$\begin{aligned}
T^{+}_1 &= \frac{\pi - 6\log 2}{4\pi}\,,\\
T^{-}_1 &= -\frac{\pi + 6\log 2}{4\pi}\,,\\
T^{+}_2 &= \frac{96K - 7\pi^2 - 36\pi\log 2 + 108(\log 2)^2}{96\pi^2}\,,\\
T^{-}_2 &= \frac{96K + 7\pi^2 - 36\pi\log 2 - 108(\log 2)^2}{64\pi^2}\,, && (4.47)\\
&\dots && (4.48)
\end{aligned}$$

4.2.2 The universal scaling function

There is no difficulty to carry out the procedure for the higher orders in ϵ. The only diffference will be that the expansion (4.33) and therefore the r.h.s. of (4.44) will contain some negative powers of ℓ. We do not go into details because the procedure is technically identical as the one formulated in [37]. The lowest orders for $\alpha_n(\epsilon)$ are:

$$\begin{aligned}
\alpha_0(\epsilon) &= 1 - \frac{1}{8}\epsilon + \dots\,, && (4.49)\\
\alpha_1(\epsilon) &= \frac{\pi - 6\log 2}{16\pi} + \frac{-96K + (7\pi - 12\log 2)(\pi + 6\log 2)}{128\pi^2}\,\epsilon + \dots\,, && (4.50)\\
\alpha_{-1}(\epsilon) &= \frac{\pi + 6\log(2)}{16\pi} + \frac{-96K - 7\pi^2 + 54\pi\log 2 + 216(\log 2)^2}{384\pi^2}\,\epsilon + \dots\,, && (4.51)\\
\alpha_2(\epsilon) &= \frac{5\left(96K - 7\pi^2 - 36\pi\log 2 + 108(\log 2)^2\right)}{1536\pi^2} + \dots\,, && (4.52)\\
\alpha_{-2}(\epsilon) &= \frac{96K + 7\pi^2 - 36\pi\log 2 - 108(\log 2)^2}{512\pi^2} + \dots\,. && (4.53)\\
& && (4.54)
\end{aligned}$$

From here we reproduce the result of [37] for the universal scaling function,

$$f(\epsilon) = \frac{1}{\epsilon} + 4\sum_{n=1}^{\infty} \epsilon^{|n|-1} n\, \alpha_n(\epsilon) \tag{4.55}$$

$$= \frac{1}{\epsilon} - \frac{3\log 2}{\pi} - \frac{K}{\pi^2}\epsilon + \dots\ . \tag{4.56}$$

5. Conclusion

We have reformulated the Beisert, Eden and Staudacher equation in terms of a functional equation obeyed by the resolvent. A similar approach was attempted, although not fully exploited, by Kotikov and Lipatov [27]. As shown recently by Basso, Korchemsky and Kotański [37], in the strong coupling perturbative regime it is possible to find the general solution as a linear combination of a set of particular functions. This is possible because, in the absence of non-perturbative terms, the resolvent possesses extra symmetries. We have shown that the “quantization condition” of [37], which allows to fix the coefficients of the linear combination order by order in the inverse coupling constant ϵ can be understood as an analyticity condition on the resolvent. The condition that the resolvent has an integrable singularity of the square root type at the points $u = \pm 1$, together with the conditions on the behavior at infinity of the resolvent are sufficient to fix the solution recursively order by order in ϵ.

Although we have not explicitly investigated the non-perturbative corrections, their source is clearly identified at the level of the resolvent. This object possesses a sequence of self-repeating cuts situated at a distance 2ϵ of one another. When $\epsilon \to 0$, the cuts condense and we are left with a single cut plus a non-perturbative term. We leave the investigation of the functional BES equation for a future work.

One of the points of technical importance in the work of [37] and in our work was to transform the BES equation into a set of two equations with the so-called undressed kernel appearing linearly. In order to perform this transformation we are led to introduce an auxiliary density. It would be interesting to know whether such a linearization is possible for the general Bethe ansatz equations for $\mathcal{N} = 4$ SYM, and if the auxiliary density can be given a physical meaning. Suggestions about the possibility that the dressed kernel originates from the elimination of an auxiliary set of Bethe roots have been made in [40, 41].

It would be interesting to check if the same method can be applied for the integral equations corresponding to other sectors of the $\mathcal{N} = 4$ gauge theory, as well as for the other limits in the $sl(2)$ sector. In particular, it would be interesting to try to reproduce the new universal scaling function predicted in [16] and [42] and computed by Roiban and Tseytlin [43] in the so-called slow long string limit.

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Figure 4: Left: Riemann surface for $g_\pm(z)$. Right: Riemann surface for $G_\pm(z)$. Dots denote branch points on the physical sheet, crosses denote the positions of the branch points on the lower sheets.

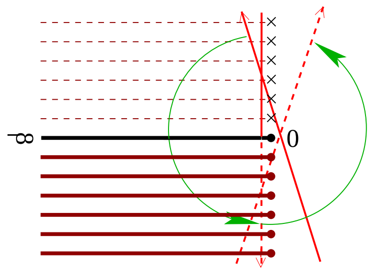

Figure 5: Deformation of the integration contour for the inverse Laplace transform of $G_\pm$.

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A. Analytical structure of $\tilde{g}_\pm$ and $\tilde{G}_\pm$

The inverse Laplace transform $\tilde{f}(\ell)$, originally defined by (4.25) for $\ell > 0$, can be analytically continued for complex values of ℓ by rotating the integration contour so that asymtotically $\Re(z\ell) = 0$ at large z.

The functions $g_\pm(z)$ have one second-order branch point at $z = 0$ on the physical sheet and an infinite sequence of equidistant branch points $z \in i\mathbb{Z}$ on the second sheet (figure 4, left). We assume that we are in the NFS limit in which the left endpoints of the branch cuts are sent to $-\infty$. We analytically continue $\tilde{g}_\pm(\ell)$ beginning from ℓ real positive by changing the phase of ℓ clockwise. The contour of integration in the definition of the inverse Laplace transform will correspondingly rotate counterclockwise. For $\ell \in -i\mathbb{R}_+$, the integration contour will lie along the real axis and above the cut of $g_\pm(z)$. We can further decrease the phase of ℓ by rotationg the contour so that half of it passes in the lower sheets of the Riemann surface. The procedure can be continued without encountering any singularity until the phase of ℓ is rotated by π

$$\ell \to e^{i\pi}\ell = -\ell \,. \tag{A.1}$$

At this point, the integration contour goes again on the imaginary axis, with the lower half now approaching a sequence of branch points on the lower sheets. Since the contour cannot be moved further, we deduce that $\tilde{g}_\pm(\ell)$ have singularities on the negative real axis.

The functions $G_\pm(z)$ have a sequence of branch points on the negative imaginary axis on the physical sheet (figure 4, right). Since $G_\pm$ do not decrease sufficiently fast at infinity, the inverse Laplace transform does not exist for $\ell > 0$. However if we rotate slightly the contour counterclockwise, as is shown in figure 5, the integral (4.25) starts to converge. In particular, it is well defined on the negative real axis, when $\ell \to e^{i\pi}\ell$. After rotating the contour by angle π, half of it passes on the second sheet, where there are no branch points below the real axis. Therefore we can continue rotating the integration contour until $\ell \to \epsilon^{2i\pi}\ell$, when we encounter the branch points on the second sheet, which are on the positive imaginary axis. The inverse Laplace transform $\tilde{G}_\pm(\ell)$ is therefore well defined for $\ell < 0$ and the singularities only occur for ℓ on the positive real axis.

B. Relation with the BKK conventions

The functions used in the present paper and those used in the paper by Basso, Korchemsky and Kotański [37] are related as follows:

$$X_{\rm here}(u) = -i \int_0^\infty dt e^{itu} Y_{\rm BKK}(t)\,,$$

with

$X_{\rm here}(u)$	$Y_{\rm BKK}(t)$
$r_+(u)$	$\gamma_-(t)$
$r_-(u)$	$\gamma_+(t)$
$\Gamma_+(u) - 2i\epsilon$	$\frac{1}{2}\Gamma_-(t)$
$\Gamma_-(u)$	$-\frac{1}{2}\Gamma_+(t)$

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